

# Stefan-Boltzmann law for massive photons

E. S. Moreira Jr.<sup>a</sup> and T. G. Ribeiro<sup>b,1</sup>

<sup>a</sup>*Instituto de Matemática e Computação, Universidade Federal de Itajubá,  
Itajubá, Minas Gerais 37500-903, Brazil*

<sup>b</sup>*Instituto de Ciências Exatas, Universidade Federal de Juiz de Fora,  
Juiz de Fora, Minas Gerais 36036-900, Brazil*

## Abstract

This paper generalizes the Stefan-Boltzmann law to include massive photons. A crucial ingredient to obtain the correct formula for the radiance is to realize that a massive photon does not travel at the speed of (massless) light. It follows that, contrary to what could be expected, the radiance is not proportional to the energy density times the speed of light.

## 1 Introduction

The understanding of the the Stefan-Boltzmann law is one of the most celebrated landmarks of modern physics. In order to derive it an important aspect is the speed at which energy leaves the blackbody cavity. If one considers the electromagnetic radiation composed of massless photons, the speed they leave the cavity is taken to be the speed of light  $c$ , leading to a formula for the radiance as being proportional to the energy density times  $c$ , i.e., the familiar Stefan-Boltzmann law [1].

The present work addresses this issue for a gas of massive photons. Formulas for the mean speed  $\bar{v}$  and the radiance  $\mathcal{R}$  are determined for arbitrary mass of the photon and at all temperatures. The corresponding high temperature (or small mass) and low temperature (or large mass) regimes are discussed. It should be noted that the Stefan-Boltzmann law for massive photons was considered previously in the literature [2]. However, the calculations were spoiled by the incorrect assumption that the massive photon's speed is  $c$ .

## 2 Speed of a massive photon

As is well known there are two equivalent ways to arrive at the thermodynamics of radiation at temperature  $T$  in a cavity of volume  $V$ . One is to consider an ensemble of quantum harmonic oscillators, and another that treats hot radiation as a Bose gas of relativistic particles with vanishing chemical potential [1]. According to the latter approach the distribution function is given by

$$f(\mathbf{p}) = \frac{2}{e^{\beta\epsilon} - 1}, \quad (1)$$

where  $\beta := 1/kT$  and  $\epsilon := pc$  is the energy of the photon of momentum  $\mathbf{p}$  (note that  $p := \|\mathbf{p}\|$ ). Now, assuming that the photon has mass  $m$  the expression for  $\epsilon$  is replaced by

$$\epsilon := \sqrt{p^2c^2 + m^2c^4}. \quad (2)$$

---

<sup>1</sup>E-mail: moreira@unifei.edu.br, tiaggribeiro@gmail.com.

In fact, the factor 2 in Eq. (1) should be replaced by 3 taking into account an additional degree of freedom associated with a nonvanishing  $m$ ; however it will be assumed that such a degree of freedom can be ignored in the thermal equilibrium of radiation and matter (see e.g. Ref. [2]).

The number of photons  $N$  in the cavity is obtained by integrating Eq. (1) over the phase space of one photon,

$$\frac{N}{V} = \int \frac{d^3p}{h^3} f(\mathbf{p}) = \frac{8\pi}{h^3} \int_0^\infty dp \frac{p^2}{e^{\beta\epsilon} - 1}, \quad (3)$$

where spherical coordinates have been used in the last equality. With the help of the geometric series and noting Eq. (2), Eq. (3) can be recast as

$$\frac{N}{V} = \frac{8\pi}{(hc)^3} \sum_{n=1}^{\infty} \int_{mc^2}^{\infty} d\epsilon \epsilon \sqrt{\epsilon^2 - m^2 c^4} e^{-n\beta\epsilon}, \quad (4)$$

where one identifies an integral representation of the modified Bessel function of the second kind  $K_2(z)$ , resulting in

$$\frac{N}{V} = \frac{m^2 c k T}{\pi^2 \hbar^3} \sum_{n=1}^{\infty} n^{-1} K_2(nx), \quad x := \frac{mc^2}{kT}. \quad (5)$$

As a simple check, considering that for very small arguments  $K_2(z)$  behaves approximately as  $2/z^2$ , by setting  $m \rightarrow 0$  in Eq. (5) it follows  $N/V = (2\zeta(3)/\pi^2)(kT/\hbar c)^3$  which is a familiar expression of blackbody radiation.

The speed of the photon  $\mathbf{v}$  and its momentum are related by  $\mathbf{p} = (\epsilon/c^2)\mathbf{v}$ , leading to (below  $v := ||\mathbf{v}||$ )

$$p = \frac{\epsilon}{c^2} v, \quad v = c \sqrt{1 - (mc^2/\epsilon)^2}, \quad (6)$$

clearly showing that  $v = c$  only if the photon is massless. The mean value of  $v$  can be obtained by averaging over the phase space with Eq. (1), i.e.,

$$\bar{v} = \frac{V}{N} \int \frac{d^3p}{h^3} v f(\mathbf{p}) = \frac{8\pi c^2 V}{N h^3} \int_0^\infty dp \frac{p^3}{\epsilon(e^{\beta\epsilon} - 1)}, \quad (7)$$

where the first expression in Eq. (6) has been used. Now, following along the same steps that led from Eq. (3) to Eq. (5) and using the formula for  $N$  in Eq. (5), one finds,

$$\bar{v} = 2c [Li_3(e^{-x}) + x Li_2(e^{-x})] \left[ x^2 \sum_{n=1}^{\infty} n^{-1} K_2(nx) \right]^{-1}, \quad (8)$$

where

$$Li_s(z) := \sum_{n=1}^{\infty} n^{-s} z^n \quad (9)$$

is the polylogarithm function [note that  $Li_s(1) = \zeta(s)$ ]. By taking  $m \rightarrow 0$  [i.e.,  $x \rightarrow 0$ , cf. Eq. (5)] in Eq. (8) it results  $\bar{v} = c$  for all values of  $T$ . However, if  $m$  is strictly nonvanishing, when

$x \gg 1$  (i.e., at low temperatures), Eq. (8) yields the following asymptotic behavior [recall that  $K_\nu(z)$  behaves approximately as  $\sqrt{\pi/2z} e^{-z}$  for large arguments],

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}, \quad (10)$$

which is the mean speed of a molecule of an ordinary nonrelativistic gas (see e.g. Ref. [1]). Thus, as  $T$  approaches the absolute zero, the massive photons disappear from the cavity [cf. Eq. (5)], and in doing so they come to rest. The plot in Fig. 1 illustrates these facts.

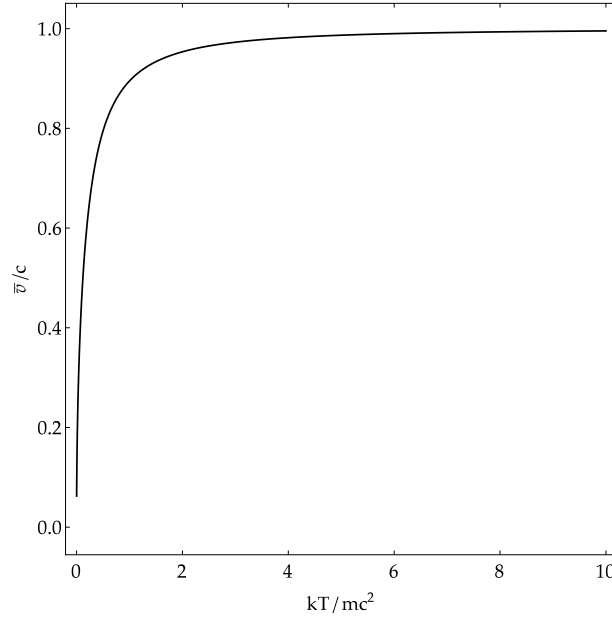


Fig. 1. Thermal behavior of the mean speed  $\bar{v}$  of a massive photon according to Eq. (8). Note that as  $T \rightarrow \infty$ ,  $\bar{v} \rightarrow c$ , and that at low temperatures  $\bar{v}$  behaves approximately as in Eq. (10).

### 3 Radiance

As  $N$  in Eq. (3), the internal energy  $U$  can also be obtained by using Eq. (1),

$$\frac{U}{V} = \int \frac{d^3p}{h^3} \epsilon f(\mathbf{p}) = \frac{8\pi}{h^3} \int_0^\infty dp \frac{p^2 \epsilon}{e^{\beta \epsilon} - 1}. \quad (11)$$

In order to make contact with the corresponding Planckian formula, it is convenient to consider  $\epsilon = \hbar\omega$  and to rewrite Eq. (11), noticing Eq. (2), resulting in

$$\frac{U}{V} = \int_{mc^2/\hbar}^\infty d\omega u(\omega, T, m), \quad (12)$$

where

$$u(\omega, T, m) := \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta \hbar \omega} - 1} \sqrt{1 - (mc^2/\hbar\omega)^2}. \quad (13)$$

By setting  $m = 0$  in Eqs. (12) and (13) the Planckian result emerges as expected. [It should be remarked that an expression for  $U$  as that for  $N$  in Eq. (5) is also available but it will not be needed here.]

The energy density  $U/V$  as given by Eqs. (12) and (13) is in agreement with Ref. [2]. According to Ref. [2] the radiance  $\mathcal{R}$  (which is defined as the amount of energy radiated per second per unit of area of an opening on the wall of the cavity) is related to the energy density as,

$$\mathcal{R} = \frac{c}{4} \frac{U}{V}. \quad (14)$$

In the following it will be shown that Eq. (14) is incorrect if the photon has a nonvanishing mass.

Consider a small opening on the wall of the cavity. The power radiated per unity of area can be obtained by calculating the flux of  $(d\Omega/4\pi)u\mathbf{v}$  through the surface of the opening (see e.g. Ref. [3]) taking into account all possible values of  $\epsilon = \hbar\omega$  and all possible directions of flight starting from inside the cavity:

$$\mathcal{R} = \int_{mc^2/\hbar}^{\infty} d\omega (uv/4), \quad (15)$$

with  $v$  and  $u$  given in Eqs. (6) and (13). It should be noticed that  $uv/4$  in Eq. (15) came about by solving

$$\int \frac{d\Omega}{4\pi} uv \cos \theta,$$

where the integration considers only one hemisphere and  $\theta$  is the angle between the straight line normal to the opening and the straight line associated with photon's flight out of the cavity. Clearly, Eq. (14) follows from Eq. (15) only for massless photons [cf. Eqs. (6) and (12)].

In a manner similar to that which led to Eq. (4), the use of the geometric series allows one to evaluate the integration in Eq. (15),

$$\mathcal{R} = \frac{3}{2(\pi c)^2 \hbar^3} (kT)^4 \left[ Li_4(e^{-x}) + x Li_3(e^{-x}) + \frac{x^2}{3} Li_2(e^{-x}) \right]. \quad (16)$$

It is instructive to consider the behavior of Eq. (16) for small mass [more precisely when  $x \ll 1$ , cf. Eq. (5)], namely

$$\mathcal{R} = \frac{\pi^2}{60\hbar^3 c^2} (kT)^4 \left[ 1 - \frac{5}{2\pi^2} \left( \frac{mc^2}{kT} \right)^2 \right], \quad \frac{mc^2}{kT} \ll 1, \quad (17)$$

where one recognizes the first term in the expression for  $\mathcal{R}$  in Eq. (17) as the Stefan-Boltzmann law of blackbody radiation. Now, noticing Eq. (9), for a nonvanishing mass and at very low temperatures Eq. (16) yields

$$\mathcal{R} = \frac{1}{2\hbar^3} \left( \frac{mc}{\pi} \right)^2 (kT)^2 e^{-mc^2/kT}, \quad \frac{mc^2}{kT} \gg 1, \quad (18)$$

showing that  $\mathcal{R}$  fades away much more quickly than its massless counterpart in the Stefan-Boltzmann law as the absolute zero of temperature is approached.

## 4 Final remarks

Before closing, it should be pointed out that the second term in the expression for  $\mathcal{R}$  in Eq. (17) is twice that in Ref. [2] and therefore the amendments proposed above do not improve much the possibility of measuring the mass of the photon by considering the deviation from the Stefan-Boltzmann law in Eq. (17) [2]. However, by progressively lowering the temperature, at some point the radiance  $\mathcal{R}$  for massive photons in Eq. (16) will differ radically from the Stefan-Boltzmann law [cf. Eq. (18)]. A word of caution is due regarding this remark though. In the context of quantum field theory at finite temperature, one learns that the use of the distribution in Eq. (1) to address radiation in a cavity of finite volume is not always a reliable approach. Roughly speaking, if the smallest dimension  $a$  of the cavity is such that  $kTa/\hbar c$  is not big enough (ideally infinite), quantum vacuum cannot be ignored, and it will play a role, resulting that Eq. (16) no longer can be trusted [4].

## Acknowledgements

This work was partially supported by the “Coordenação de Aperfeiçoamento de Pessoal de Nível Superior” (CAPES) and the “Fundação de Amparo à Pesquisa do Estado de Minas Gerais” (FAPEMIG).

## References

- [1] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Elsevier, Oxford, 1980), p. 183
- [2] J. Torres-Hernández, “Photon mass and blackbody radiation,” *Phys. Rev. A* **32**, 623 (1985)
- [3] T. R. Cardoso and A. S. de Castro, “The blackbody radiation in a D-dimensional universes,” *Rev. Bras. Ens. Fis.* **27**, 559 (2005)
- [4] L. S. Brown and G. J. Maclay, “Vacuum Stress between Conducting Plates: An Image Solution,” *Phys. Rev.* **184**, 1272 (1969)